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Convergence Theorems for Best Approximations in a Nonreflexive Banach Space

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In this paper, we study continuity properties of the mapping $P: (x, A) \to P_A(x)$ in a nonreflexive Banach space where P_A is the metric projection onto A. Our results extend the existing convergence theorems on the best approximations in a reflexive Banach space to nonreflexive Banach spaces by using Wijsman convergence of sets. © 1998 Academic Press

1. INTRODUCTION

The problem of continuity for the mapping $A \to P_A(x)$ was first considered by Brosowski, Deutsch, and Nürnberger [2]. They considered a family $\{A_t\}_{t \in T}$ of subsets of a normed linear space X parametrized by a topological space T and studied the continuity of $t \to P_{A_t}(x)$. In 1984, Tsukada [12] discussed the above problem but with a nonparametrized method. He proved that if the Banach space X is reflexive strictly convex, then when the closed convex sets $\{A_n\}$ converge to A in the Mosco sense it is implied that $\{P_{A_n}(x)\}$ weakly converges to $P_A(x)$ for each $x \in X$. Moreover if X has the property (H), the convergence is in the norm. Papageorgion and Kandilakis [8] generalized the work of Tsukada and studied the convergence of ε-approximations. Recently, the author and Nan [14] gave a further description on the convergence of best approximations. In these convergence theorems for best approximations, one standard condition is that the Banach space X is reflexive. The purpose of this paper is to establish new convergence results for best approximations in a nonreflexive Banach space. For this purpose, we need a notion of strong Wijsman convergence for the sequence of nonempty sets. Under the assumption of the strong Wijsman convergence we give two continuity

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theorems of the mapping $P\colon (x, A) \to P_A(x)$ in nonreflexive spaces. Moreover we prove that in a reflexive Banach space, if the sequence of closed convex subsets $\{A_n\}$ converges to A in the Mosco sense, then $\{A_n\}$ converges to A in the strong Wijsman sense. As a consequence, our results are natural extensions of the existing results on the continuity of metric projections $P_A(x)$ (as a set-valued mapping of (x, A)) in a reflexive Banach space [12–16] to nonreflexive Banach spaces.

2. NOTATIONS AND DEFINITIONS

Let $\{A_n\}$ be a sequence of nonempty subsets of a Banach space. Define the weak limit superior of the sequence $\{A_n\}$ to be the set

$$w\text{-}\overline{\lim_n}\, A_n = \{x \in X : x = w\text{-}\lim_n x_{n_k},\ x_{n_k} \in A_{n_k},\ k \geqslant 1\}$$

and the strong limit inferior of $\{A_n\}$ to be the set

$$s\text{-}\underline{\lim_n}\, A_n = \{x \in X : x = \lim_n x_n,\ x_n \in A_n,\ n \geqslant 1\}.$$

A sequence of nonempty subsets $\{A_n\}$ of a Banach space X is said to converge to a set A in the Mosco sense if $s\text{-}\underline{\lim}_n A_n = w\text{-}\overline{\lim}_n A_n = A$. We will write $\lim_{M,n} A_n = A$ or $A_n \xrightarrow{M} A$.

A sequence of nonempty subsets $\{A_n\}$ of a Banach space X is said to converge to a set A in the Wijsman sense if $\lim_n d(x, A_n) = d(x, A)$ for each x in X, where $d(x, A) = \inf_{y \in A} \|x - y\|$. We will write $\lim_{W,n} A_n = A$ or $A_n \xrightarrow{W} A$.

Let X be a Banach space and A be a subset of X. $P_A(x)$ denotes the set of all best approximations to x from A; i.e., $P_A(x) = \{y \in A : \|x - y\| = d(x, A)\}$. The set A is proximinal (resp. Chebyshev) if $P_A(x)$ contains at least (resp. exactly) one point for each x in X.

For any $\varepsilon > 0$, we say that an element z in A is an ε-approximation of x in A if $\|x - z\| \leqslant d(x, A) + \varepsilon$. We will denote the set of all ε-approximations by $P_A^{\varepsilon}(x)$.

Let $U(X)$ denote the closed unit ball of a Banach space X and let $S(X)$ denote the unit sphere of X. Also $\mathrm{co}(B)$ means the convex hull of B.

A sequence of nonempty subsets $\{A_n\}$ of a Banach space X is said to converge to a set A in the strong Wijsman sense, if for each fixed x in X and any $\varepsilon_n \to 0^+$ as $n \to \infty$, we have $\lim_n \|x - u_n\| = d(x, A)$ whenever $u_n \in \mathrm{co}(\bigcup_{k=n}^{\infty} P_{A_k}^{\varepsilon_k}(x))$ for $n \geqslant 1$. We will write $s\text{-}\lim_{W,n} A_n = A$ or $A_n \xrightarrow{sW} A$.

Clearly, $\{A_n\}$ converges to A in the Wijsman sense if and only if for each x in X and any $\varepsilon_n \to 0^+$ as $n \to \infty$, we have $\lim_n \|x - a_n\| = d(x, A)$ whenever $a_n \in P_{A_n}^{\varepsilon_n}(x)$ for $n \geqslant 1$. Furthermore, $s\text{-}\lim_n {}_W A_n = A$ implies $\lim_n {}_W A_n = A$.

Let $\mathscr{C}$ be the set of all proximinal subsets of X.

The mapping $x \to P_A(x)$ is said to be (weakly) upper semicontinuous if for each fixed x in X and any (weakly) open set $W \supset P_A(x)$, there exists a neighborhood U of x such that $P_A(U) \subset W$.

We say that the mapping $P: (x, A) \to P_A(x)$ is (weakly) upper semicontinuous at (x, A) in the strong Wijsman sense if, given a (weakly) open set W containing $P_A(x)$, any $x_n \to x$, and any sequence $\{A_n\} \subset \mathscr{C}$ with $A_n \xrightarrow{sW} A$, there exists a corresponding integer N such that $P_{A_n}(x_n) \subset W$ for all $n \geqslant N$.

The mapping $P: (x, A) \to P_A(x)$ is called (weakly) upper semicontinuous in the strong Wijsman sense if it is (weakly) upper semicontinuous at each (x, A) in the strong Wijsman sense.

In [13], a Banach space X is said to have the proper (C-I) (resp. (C-II) or (C-III)), if $x \in S(X)$, $\{x_n\}$ in $U(X)$ with the property that for each $\delta > 0$ there exists an integer $N(\delta)$ such that

$$\operatorname{co}(\{x\} \cup \{x_n : n \geqslant N(\delta)\}) \cap (1 - \delta)\, U(X) = \varnothing,$$

then $\lim_n \|x_n - x\| = 0$ (resp. $\{x_n\}$ is relatively compact or $\{x_n\}$ is relatively weakly compact).

We say that a Banach space X has the property (H), if for sequences on the unit sphere of X weak convergence is equivalent to norm convergence.

We also mention some convexity properties for X. These definitions can be found in [4, 13, 15].

A Banach space X is locally nearly uniformly convex (LNUC) if for each $x \in S(X)$ and each $\varepsilon > 0$, there is a positive constant $\delta = \delta(x, \varepsilon)$ such that for all sequences $\{x_n\}$ in $U(X)$ with $\inf\{\|x_n - x_m\| : n \neq m\} > \varepsilon$, we have

$$\operatorname{co}(\{x\} \cup \{x_n : n \geqslant 1\}) \cap (1 - \delta)\, U(X) \neq \varnothing.$$

A Banach space X is compactly locally fully k rotund (CL-kR), if for any sequence $\{x_n\}$ in $U(X)$ and for any $x \in S(X)$ with

$$\lim_{n_1, \ldots, n_k \to \infty} \|x + x_{n_1} + \cdots + x_{n_k}\| = k + 1,$$

then $\{x_n\}$ is relatively compact.

Replacing the phrase "$\{x_n\}$ is relatively compact" by "$\{x_n\}$ is relatively weakly compact," X is called a (WCL-kR) space.

Finally, we give some results concerned with the strong Wijsman convergence of sets and the properties (C-II) and (C-III). They will be used in Section 3.

PROPOSITION 2.1. *Let X be a reflexive Banach space. If A_n and A are nonempty closed convex subsets of X, then* $\lim_{n M} A_n = A$ *implies* $s\text{-}\lim_{n W} A_n = A$.

Proof. We prove the proposition by contradiction. Assume the contrary, that there exists an x_0 and sequences $\{\varepsilon_n\}$, $\{u_{n_i}\}$ with $u_{n_i} \in \operatorname{co}(\bigcup_{k=n_i}^{\infty} P_{A_k}^{\varepsilon_k}(x_0))$ such that $\varepsilon_n \to 0^+$ as $n \to \infty$, but $|\,\|x_0 - u_{n_i}\| - d(x_0, A)| > \eta > 0$ for $i \geqslant 1$. Let $u_{n_i} = \sum_{j=1}^{m_i} \lambda_j^{(i)} a_j^{(i)}$ where $\lambda_j^{(i)} \geqslant 0$, $\sum_{j=1}^{m_i} \lambda_j^{(i)} = 1$, $a_j^{(i)} \in P_{A_k}^{\varepsilon_k}(x_0)$ for some $k \geqslant n_i$, $j = 1, 2, ..., m_i$, $i \geqslant 1$. Then

$$\begin{aligned} \|x_0 - u_{n_i}\| &\leqslant \sum_{j=1}^{m_i} \lambda_j^{(i)} \|x_0 - a_j^{(i)}\| \\ &\leqslant \sup_{k \geqslant n_i} \{d(x_0, A_k) + \varepsilon_k\} \qquad \text{for all } i. \end{aligned}$$

It is known from [1] that for a reflexive Banach space, $\lim_{n M} A_n = A$ implies $\lim_{n W} A_n = A$. It follows that $\{u_{n_i}\}$ and $\{a_j^{(i)} : j = 1, 2, ..., m_i, i \geqslant 1\}$ are bounded and $\overline{\lim_i} \|x_0 - u_{n_i}\| \leqslant d(x_0, A)$. Thus we can select a subsequence, denoted by $\{u_{n_i}\}$ again, such that

$$\lim_i \|x_0 - u_{n_i}\| < \gamma < d(x_0, A).$$

By the assumption that X is reflexive, $\{u_{n_i}\}$ has a weakly convergent subsequence. Without loss of generality, we may assume $w\text{-}\lim_i u_{n_i} = y$. If $y \in A$, then $d(x_0, A) \leqslant \|x_0 - y\| \leqslant \underline{\lim}_i \|x_0 - u_{n_i}\| < \gamma < d(x_0, A)$. This is impossible. Therefore $y \notin A$. By the strong separation theorem, there is an $x_0^* \in X^*$ such that

$$x_0^*(y) > \alpha > \sup_{z \in A} x_0^*(z).$$

Since $y = w\text{-}\lim_i u_{n_i}$, we can assume that $x_0^*(u_{n_i}) > \alpha$ for all i. Since $u_{n_i} = \sum_{j=1}^{m_i} \lambda_j^{(i)} a_j^{(i)}$, there exists an index j_i, $1 \leqslant j_i \leqslant m_i$, such that $x_0^*(a_{j_i}^{(i)}) > \alpha$. Observe that $\{a_{j_i}^{(i)}\}$ is bounded. Hence $\{a_{j_i}^{(i)}\}$ has a weakly convergent subsequence; for convenience, assume $z_0 = w\text{-}\lim_i a_{j_i}^{(i)}$. Then $x_0^*(z_0) = \lim_i x_0^*(a_{j_i}^{(i)}) \geqslant \alpha$. However, by the assumption that $\lim_{n M} A_n = A$, we obtain $z_0 \in w\text{-}\overline{\lim_n} A_n = A$. Consequently, $x_0^*(z_0) \leqslant \sup_{z \in A} x_0^*(z) < \alpha$. This is a contradiction. Therefore $s\text{-}\lim_{n W} A_n = A$.

PROPOSITION 2.2. *If $\{A_n\}$ is an increasing sequence of nonempty convex subsets of a Banach space X, then* $s\text{-}\lim_{n W} A_n = \overline{\bigcup_{n=1}^{\infty} A_n}$. *Moreover,* $\lim_{n M} A_n = \overline{\bigcup_{n=1}^{\infty} A_n}$.

Proof. Let $x \in X$. We first prove that $\lim_n d(x, A_n) = d(x, \overline{\bigcup_{n=1}^{\infty} A_n})$. Since $\{A_n\}$ is increasing, obviously $\lim_n d(x, A_n) \geqslant d(x, \overline{\bigcup_{n=1}^{\infty} A_n})$. On the other hand, if $y \in \overline{\bigcup_{n=1}^{\infty} A_n}$, we can choose $y_k \in \bigcup_{n=1}^{\infty} A_n$ such that $\|y - y_k\| < 1/k$ for $k \geqslant 1$. Since $A_1 \subset A_2 \subset ...$, we have $y_k \in A_n$ for all n large enough. Thus,

$$\|x - y\| \geqslant \|x - y_k\| - \|y - y_k\| \geqslant d(x, A_n) - 1/k$$

for all n large enough and all k, which implies that

$$d\left(x, \overline{\bigcup_{n=1}^{\infty} A_n}\right) \geqslant \lim_n d(x, A_n).$$

Therefore $\lim_n d(x, A_n) = d(x, \overline{\bigcup_{n=1}^{\infty} A_n})$.

Now we prove $s\text{-}\lim_{W\,n} A_n = \overline{\bigcup_{n=1}^{\infty} A_n}$. Let $x \in X$, $\varepsilon_n \to 0^+$ as $n \to \infty$ and let $u_n \in \operatorname{co}(\bigcup_{k=n}^{\infty} P_{A_k}^{\varepsilon_k}(x))$, $n \geqslant 1$. As in the proof of Proposition 2.1, we can derive that

$$\|x - u_n\| \leqslant \sup_{k \geqslant n} \{d(x, A_k) + \varepsilon_k\} \tag{1}$$

for all n. Since $\{A_n\}$ is an increasing sequence of convex sets, clearly $\operatorname{co}(\bigcup_{k=n}^{\infty} A_k) = \bigcup_{k=n}^{\infty} A_k$. Hence $u_n \in A_k$ for all k large enough. This means that

$$\|x - u_n\| \geqslant d(x, A_k) \tag{2}$$

for all k large enough. From (1), (2), and $\lim_n d(x, A_n) = d(x, \overline{\bigcup_{n=1}^{\infty} A_n})$, it follows that $\lim_n \|x - u_n\| = d(x, \overline{\bigcup_{n=1}^{\infty} A_n})$. So $s\text{-}\lim_{W\,n} A_n = \overline{\bigcup_{n=1}^{\infty} A_n}$.

By using the fact that $\lim_n d(x, A_n) = d(x, \overline{\bigcup_{n=1}^{\infty} A_n})$ and $\overline{\bigcup_{n=1}^{\infty} A_n}$ is convex, it is easy to verify that $w\text{-}\overline{\lim_n} A_n \subset \overline{\bigcup_{n=1}^{\infty} A_n} \subset s\text{-}\underline{\lim_n} A_n$. Hence $\lim_{M\,n} A_n = \overline{\bigcup_{n=1}^{\infty} A_n}$.

In [13], the following statements have been proved: (i) Every (LNUC) or (CL-kR) Banach space has the property (C-II); (ii) Every (WCL-kR) space has the property (C-III).

For a reflexive Banach space, we have the following proposition about the property (C-II).

PROPOSITION 2.3. *Let X be a reflexive Banach space and let X have the property* (H). *Then X has the property* (C-II).

Proof. Let $x \in S(X)$ and $\{x_n\} \subset U(X)$ be such that for each $\delta > 0$, there exists an integer $N(\delta)$ satisfying

$$\operatorname{co}(\{x\} \cup \{x_n : n \geqslant N(\delta)\}) \cap (1 - \delta)\, U(X) = \varnothing.$$

Since X is reflexive, $\{x_n\}$ is relatively weakly compact. Suppose that $\{x_{n_i}\}$ is a weakly convergent subsequence of $\{x_n\}$ and $w\text{-}\lim_i x_{n_i} = x_0$. It is known that any closed convex set is weakly closed. So $x_0 \in \overline{\text{co}}(\{x\} \cup \{x_n : n \geqslant N(\delta)\})$. Hence $\|x_0\| \geqslant 1 - \delta$ for each $\delta > 0$. Therefore we have $\|x_0\| = 1$. Because X has the property (H), it follows that $x_{n_i} \to x_0$. This proves that $\{x_n\}$ is relatively compact.

3. CONVERGENCE OF BEST APPROXIMATIONS IN NONREFLEXIVE BANACH SPACES

Let X be a Banach space and X^* be the dual space of X.

LEMMA 3.1. *Let X be a Banach space with the property* (C-II), $A_n \subset X$ $(n \geqslant 1)$, *A in $\mathscr{C}$, and* $s\text{-}\lim_n{}_W A_n = A$. *If $x_n \to x$ and $y_n \in P_{A_n}(x_n)$ $(n \geqslant 1)$, then $\{y_n\}$ is relatively compact and its convergent subsequence converges to an element of $P_A(x)$.*

Proof. Let $x \in X$, $x_n \to x$, $y_n \in P_{A_n}(x_n)$, $n = 1, 2, \ldots$. Clearly, $d(x, A_n) \leqslant \|x - y_n\| \leqslant \|x - x_n\| + \|x_n - y_n\| = \|x - x_n\| + d(x_n, A_n) \leqslant \|x - x_n\| + |d(x_n, A_n) - d(x, A_n)| + d(x, A_n)$. By [10] or [12, Remark, 302], for any nonempty subset A of X, $d(\cdot, A)$ is uniformly continuous and $|d(x, A) - d(y, A)| \leqslant \|x - y\|$ for any $x, y \in X$. So we obtain that

$$d(x, A_n) \leqslant \|x - y_n\| \leqslant 2\,\|x - x_n\| + d(x, A_n) \tag{3}$$

for all n. Since $s\text{-}\lim_n{}_W A_n = A$ implies $\lim_n{}_W A_n = A$, we derive from (3) that

$$\lim_n \|x - y_n\| = d(x, A). \tag{4}$$

Fix $z \in P_A(x)$. Since $\lim_n{}_W A_n = A$, so $\lim_n d(z, A_n) = d(z, A) = 0$. Thus there exists a sequence $z_n \in A_n$, $n \geqslant 1$, such that $\lim_n \|z_n - z\| = 0$. Let $\varepsilon_n = 2\,\|x - x_n\| + \|z - z_n\| + |d(x, A_n) - d(x, A)|$, and $\alpha_n = d(x, \text{co}(\bigcup_{k=n}^{\infty} P_{A_k}^{\varepsilon_k}(x)))$. By (3), $\|x - y_n\| \leqslant d(x, A_n) + \varepsilon_n$. This, along with $y_n \in A_n$, implies $y_n \in P_{A_n}^{\varepsilon_n}(x)$. Therefore $\alpha_n \leqslant \|x - y_n\|$. Since $\{\alpha_n\}$ is increasing, by (4) we obtain $\lim_n \alpha_n \leqslant d(x, A)$. For each n, select $u_n \in \text{co}(\bigcup_{k=n}^{\infty} P_{A_k}^{\varepsilon_k}(x))$ with $\|x - u_n\| < d(x, \text{co}(\bigcup_{k=n}^{\infty} P_{A_k}^{\varepsilon_k}(x))) + 1/n$. It follows from $s\text{-}\lim_n{}_W A_n = A$ that $\lim_n \|x - u_n\| = d(x, A)$. This shows that

$$\lim_n \alpha_n = d(x, A). \tag{5}$$

Observe that $\|x - z_n\| \leqslant \|x - z\| + \|z - z_n\| \leqslant \|z - z_n\| + |d(x, A_n) - d(x, A)| + d(x, A_n) \leqslant d(x, A_n) + \varepsilon_n$ for all n. Therefore $z_n \in P^{\varepsilon_n}_{A_n}(x)$ for $n \geqslant 1$. Suppose $d(x, A) \neq 0$. Given $\delta > 0$, in view of (5) and $\lim_n \|z - z_{zn}\| = 0$, there exists an integer $N_1(\delta)$ such that when $n \geqslant N_1(\delta)$, we have

$$\alpha_n \geqslant d(x, A)(1 - \delta/4), \qquad \|z - z_n\| \leqslant d(x, A) \cdot \delta/4.$$

Further, if $\lambda_i \geqslant 0$, $i = 0, 1, ..., m$, $\sum_{i=0}^m \lambda_i = 1$, and $N_1(\delta) \leqslant n_0 \leqslant n_1 \leqslant \cdots \leqslant n_m$, we get that $\|x - (\lambda_0 z + \lambda_1 y_{n_1} + \cdots + \lambda_m y_{n_m})\| \geqslant \|x - (\lambda_0 z_{n_0} + \lambda_1 y_{n_1} + \cdots + \lambda_m y_{n_m})\| - \lambda_0 \|z - z_{n_0}\| \geqslant \alpha_{n_0} - \lambda_0 \|z - z_{n_0}\| \geqslant d(x, A)(1 - \delta/2)$. Thus,

$$\left\| \lambda_0 \frac{x - z}{d(x, A)} + \cdots + \lambda_m \frac{x - y_{n_m}}{d(x, A)} \right\| \geqslant 1 - \delta/2.$$

By $\lim_n d(x, A_n) = d(x, A)$ and (4), there exists an integer $N(\delta)$ ($\geqslant N_1(\delta)$) such that

$$\left| \frac{1}{d(x, A)} - \frac{1}{d(x, A_n) + 2\|x - x_n\|} \right| \sup_k \|x - y_k\| < \delta/2$$

for $n \geqslant N(\delta)$. Thus if $N(\delta) \leqslant n_1 \leqslant \cdots \leqslant n_m$, we get

$$\begin{aligned} &\left\| \lambda_0 \frac{x - z}{d(x, A)} + \lambda_1 \frac{x - y_{n_1}}{d(x, A_{n_1}) + 2\|x - x_{n_1}\|} \right. \\ &\qquad \left. + \cdots + \lambda_m \frac{x - y_{n_m}}{d(x, A_{n_m}) + 2\|x - x_{n_m}\|} \right\| \\ &\geqslant \left\| \lambda_0 \frac{x - z}{d(x, A)} + \lambda_1 \frac{x - y_{n_1}}{d(x, A)} + \cdots + \lambda_m \frac{x - y_{n_m}}{d(X, A)} \right\| \\ &\qquad - \sum_{i=1}^m \lambda_i \left| \frac{1}{d(x, A)} - \frac{1}{d(x, A_{n_i}) + 2\|x - x_{n_i}\|} \right| \|x - y_{n_i}\| \\ &> 1 - \delta/2 - \delta/2 = 1 - \delta. \end{aligned}$$

This shows that for each fixed $\delta > 0$, there exists an integer $N(\delta)$ such that

$$\begin{aligned} &\text{co}\left(\left\{ \frac{x - z}{d(x, A)} \right\} \cup \left\{ \frac{x - y_n}{d(x, A_n) + 2\|x - x_n\|} : n \geqslant N(\delta) \right\} \right) \\ &\qquad \cap (1 - \delta)\, U(X) = \varnothing. \end{aligned} \tag{6}$$

By (3) and the assumption that X has the property (C-II), we obtain that $\{(x - y_n)/(d(x, A_n) + 2\|x - x_n\|)\}$ is relatively compact. Further $\{y_n\}$ is relatively compact. If $d(x, A)=0$, from (4) it follows that $\{y_n\}$ is a convergent sequence. Of course it is relatively compact. Let $y_{n_i} \to y$. Observe that $d(y, A_{n_i}) \leqslant \|y-y_{n_i}\| \to 0$ and $d(y, A)=\lim_i d(y, A_{n_i})$, so $d(y, A)=0$. Since any proximinal set is closed, hence $y \in A$. By (4) we have $\|x-y\| = \lim_i \|x-y_{n_i}\| = d(x, A)$; thus $y \in P_A(x)$.

THEOREM 3.1. *Let X be a Banach space with the property* (C-II), *and $A \in \mathscr{C}$. Then the mapping P: $(x, A) \to P_A(x)$ is upper semicontinuous in the strong Wijsman sense.*

Proof. Assume the contrary, that Theorem 3.1 is not true. Then for some (x_0, A_0) and a norm open set $W_0 \supset P_{A_0}(x_0)$ and corresponding convergent sequences $x_n \to x_0$, $s\text{-}\lim_n{}_W A_n = A_0$, we can choose a subsequence $\{n_i\}$ of $\{n\}$ such that $P_{A_{n_i}}(x_{n_i}) \not\subset W_0$ for all i. Let $y_{n_i} \in P_{A_{n_i}}(x_{n_i}) \backslash W_0$, $i \geqslant 1$. By Lemma 3.1 it follows that $\{y_{n_i}\}$ has a subsequence which converges to an element of $P_{A_0}(x_0)$. This means that there exists $y_{n_i} \in W_0$ for some i large enough. This is impossible. This completes the proof of Theorem 3.1.

An immediate consequence of Theorem 3.1 is the following corollary.

COROLLARY 3.1 [13]. *Let X be a Banach space with the property* (C-II) *and A be a proximinal convex subset in X. Then the mapping $x \to P_A(x)$ is upper semicontinuous.*

Using Theorem 3.1 and Propositions 2.1 and 2.3 we derive the following corollaries.

COROLLARY 3.2 [14]. *Let X be a reflexive Banach space with the property* (H). *If A_n $(n \geqslant 1)$ and A are closed convex subsets of X with $\lim_n{}_M A_n = A$, then for each x in X and all $\{x_n\}$ we have $\lim_n(\sup_{y \in P_{A_n}(x_n)} d(y, P_A(x)))=0$ whenever $x_n \to x$.*

COROLLARY 3.3 [12]. *Let X be a strictly convex reflexive Banach space with the property* (H). *If A_n $(n \geqslant 1)$ and A are closed convex subsets of X with $\lim_n{}_M A_n = A$, then $\lim_n \|P_{A_n}(x) - P_A(x)\| = 0$ for all x in X.*

LEMMA 3.2. *Let X be a Banach space with the property* (C-III), *$A_n \subset X$ $(n \geqslant 1)$, A in $\mathscr{C}$ and $s\text{-}\lim_n{}_W A_n = A$. If $x_n \to x$ and $y_n \in P_{A_n}(x_n)$ $(n \geqslant 1)$ then $\{y_n\}$ is relatively weakly compact. If in addition $\lim_n{}_M A_n = A$ then the weakly convergent subsequence of $\{y_n\}$ converges to an element of $P_A(x)$ weakly.*

Proof. The proof of the first part is the same as Lemma 3.1. Now we prove the second part. Let $\{y_{n_i}\}$ be a weakly convergent subsequence of $\{y_n\}$ and $w\text{-}\lim_i y_{n_i} = y$. Since $\lim_{M,n} A_n = A$, so $y \in A$. By the weak lower semicontinuity of the norm and (4) we can write that

$$\|x - y\| \leqslant \underline{\lim}_i \|x - y_{n_i}\| = d(x, A).$$

But recall that $y \in A$. So $y \in P_A(x)$.

A result of Papageorgion and Kandilakis [8, Proposition 2.2] tells us that in a general Banach space, if $\lim_{M,n} A_n = A$ then $w\text{-}\overline{\lim}_n P_{A_n}(x) \subset P_A(x)$ for all x in X. Their proof also applies to the following case: If $\lim_{M,n} A_n = A$ and $x_n \to x$, then $w\text{-}\overline{\lim}_n P_{A_n}(x_n) \subset P_A(x)$. Further, the second part of Lemma 3.2 can be derived immediately by Proposition 2.2 of [8].

If $s\text{-}\lim_{W,n} A_n = \lim_{M,n} A_n = A$, we will write $A_n \xrightarrow{sW-M} A$. Following weak upper semicontinuity in the strong Wijsman sense, we can similarly introduce the concept of weak upper semicontinuity in the strong Wijsman–Mosco sense for the mapping $P: (x, A) \to P_A(x)$.

THEOREM 3.2. *Let X be a Banach space with the property* (C-III), *and A in $\mathscr{C}$. Then the mapping $P: (x, A) \to P_A(x)$ is weakly upper semicontinuous in the strong Wijsman–Mosco sense.*

If, in addition X is strictly convex and A is convex, then the mapping $P: (x, A) \to P_A(x)$ is weakly continuous in the strong Wijsman sense.

Proof. The weak upper semicontinuity of P in the strong Wijsman–Mosco sense can be proved by Lemma 3.2. The proof proceeds as in Theorem 3.1.

Next we prove the weak continuity of P in the strong Wijsman sense. Let $u \in S(X)$ and $\{u_n\}$ in $U(X)$ with the property that for each $\delta > 0$ there exists an integer $N(\delta)$ such that

$$\operatorname{co}(\{u\} \cup \{u_n : n \geqslant N(\delta)\}) \cap (1 - \delta)\, U(X) = \varnothing.$$

We will prove $w\text{-}\lim_n u_n = u$. Since X has the property (C-III), $\{u_n\}$ has a weakly convergent subsequence. Let $w\text{-}\lim_i u_{n_i} = u_0$. Since u_0 is in $\overline{\operatorname{co}}(\{u_n : n \geqslant N(\delta)\})$, so $\frac{1}{2}(u + u_0) \in \overline{\operatorname{co}}(\{u\} \cup \{u_n : n \geqslant N(\delta)\})$ for every $\delta > 0$. Therefore we obtain that $u_0 \in S(X)$ and $\|\frac{1}{2}(u + u_0)\| = 1$. From the strict convexity of X it follows $u = u_0$; thus, $w\text{-}\lim_n u_n = u$.

Now let $x_n \to x$ and $s\text{-}\lim_{W,n} A_n = A$, where A_n $(n \geqslant 1)$ and A are proximinal convex subsets of X. If $y_n \in P_{A_n}(x_n)$ for every n and $z \in P_A(x)$, from the proof of Lemma 3.1 we know that when $x \notin A$, then for each fixed $\delta > 0$ there is an integer $N(\delta)$ such that $\mathrm{co}(\{(x-z)/d(x, A)\} \cup \{(x-y_n)/(d(x, A_n)+2\|x-x_n\|) : n \geqslant N(\delta)\}) \cap (1-\delta)\ U(X) = \varnothing$. Consequently, $w\text{-}\lim_n y_n = z \in P_A(x)$. If $x \in A$, by (4) in Lemma 3.1 we have $y_n \to x \in P_A(x)$. Observe that if X is strictly convex and A is a proximinal convex subset, then A is a Chebyshev set. Therefore $w\text{-}\lim_n P_{A_n}(x_n) = P_A(x)$.

Recall that a reflexive Banach space has the property (C-III) and $\lim_n{}_M A_n = A$ implies $s\text{-}\lim_n{}_W A_n = A$. Thus we have the following corollaries of Theorem 3.2.

COROLLARY 3.4 [12]. *Let X be a reflexive and strictly convex Banach space, A_n $(n \geqslant 1)$ and let A be closed convex subsets of X with $\lim_n{}_M A_n = A$. Then $w\text{-}\lim_n P_{A_n}(x) = P_A(x)$ for each x in X.*

COROLLARY 3.5 [13]. *Let X be a Banach space with the property* (C-III) *and A be a proximinal convex subset in X. Then the mapping $x \to P_A(x)$ is weakly upper semicontinuous.*

Remark 3.1. Observe that a Banach space X has the property (C-I) if and only if X has the property (C-II) and X is strictly convex. By Theorem 3.1 it is clear that if X has the property (C-I), A_n $(n \geqslant 1)$ and A are proximinal convex sets with $s\text{-}\lim_n{}_W A_n = A$, then for each x in X, $\lim_n \|P_{A_n}(x_n) - P_A(x)\| = 0$ whenever $x_n \to x$.

Remark 3.2. Theorem 3.1 is true for (LNUC) or (CL-kR) and Theorem 3.2 is true for (WCL-kR).

Remark 3.3. In [11], F. Sullivan introduced the locally k uniformly rotund (Lk-UR) spaces. He proved that if M is a Chebyshev subspace of the (L2-UR) space, then the mapping $x \to P_M(x)$ is norm continuous. In 1985, Yu Xintai [16] extended Sullivan's theorem [11] to (Lk-UR) spaces. From Theorem 1 of [6] it is known that every (Lk-UR) space is (CL-kR). According to Remark 3.2 we get that Theorem 3.1 is true replacing the property (C-II) by (Lk-UR). Consequently, our results generalize the continuity theorems of [11, 16].

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